# GENERAL METHOD

OF

# SOLVING EQUATIONS OF ALL DEGREES;

*APPLIED PARTICULARLY TO*

## EQUATIONS OF THE SECOND, THIRD, FOURTH, AND FIFTH DEGREES.

BY

OLIVER BYRNE,

INVENTOR OF DUAL ARITHMETIC, A NEW ART; AND
THE CALCULUS OF FORM, A NEW SCIENCE.

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# PREFACE.

THE chief business of the science of algebra is to evolve the value of unknown quantities from algebraical expressions termed *equations*, in which known and unknown quantities are involved, bearing to each other given relations which can be expressed numerically. It is well known that, to establish general formulæ by which the numerical values of the roots of equations of all degrees might be determined has baffled the most ardent exertions of the ablest analysts, and that the numerical methods employed to effect the same purpose are so laborious that their practical application is almost impossible. Notwithstanding, the general method of solving equations of all degrees, which I have established and illustrated in the following pages, is general, easily applied, and pre-eminently practical, and is, without doubt, the greatest acquisition that the science of algebra has received, for without such general method the science would be incomplete and lack one of its greatest requirements.

To be able to apply this method it is necessary to have a knowledge of the art and science of Dual Arithmetic which I invented and developed in five volumes lately published. Yet, for those who have not investigated Dual Arithmetic, it may be necessary to state, that any two of the three corresponding numbers (*Natural number*), (*Dual number*), (Dual Logarithm), may be almost instantly found, the remaining one being given; and that too, without the use of tables, by easy independent and direct processes. A dual number is written thus:—

$$\downarrow u_1,\ 'u_2\ u_3,\ u_4,\ u_5,\ 'u_6\ .\ .\ .\ .\ 'u_n, \quad (A);$$

$u_1$, $u_3$, $u_4$, $u_5$, are dual digits of the ascending branch in the 1st, 3rd, 4th, and 5th positions after the arrow, marked by numerals and commas to the right *below*, $'u_2$ $'u_6$ are dual digits of the descending branch in the 2nd and 6th positions after the arrow, marked by numerals and commas to the left *above*. This dual number represents the continued product of $(1{\cdot}1)^{u_1}$; $({\cdot}99)^{u_2}$; $(1{\cdot}001)^{u_3}$; $(1{\cdot}0001)^{u_4}$; $(1{\cdot}00001)^{u_5}$; and $({\cdot}999999)^{u_6}$. The consecutive bases of the ascending branch are, 1·1; 1·01; 1·001; 1·0001, &c.; and those of the descending ·9; ·99; ·999; ·9999, &c. The dual digits of any dual number (A), may be made to assume an immense number of values without altering the corresponding natural number; and each of such dual numbers, corresponding to the same natural number, may be reduced to a constant number in the $n$th position, leaving a zero in every other position; this constant number in the $n$th position is termed a dual logarithm of that position. The dual logarithm of the natural number $a$ is written, $\downarrow,(a)$; $\downarrow,(y+\frac{1}{y})$ represents the dual logarithm of $y+\frac{1}{y}$; and so on; the comma being placed near the barb of the arrow.

In the same manner $\downarrow, u_1, u_2, 'u_3$, indicates the dual logarithm of the dual number $\downarrow u_1, u_2, 'u_3$. The accomplished mathematician must not consider my minute discussions of simple elementary propositions unnecessary, for it is my design that this method of solving equations of all degrees may be readily acquired by any student who understands the elements of algebra and common arithmetic.

OLIVER BYRNE.

# GENERAL METHOD

OF

# SOLVING EQUATIONS OF ALL DEGREES.

In supposing $w$, in (1), to have a continuous range of numerical values, no new result is obtained by imagining $v$ to be a proper fraction of the form $\frac{1}{z}$; for then (1) becomes (2).

$$v + \frac{1}{v} = w, \ (1);$$

$$z + \frac{1}{z} = w, \ (2).$$

Hence it is unnecessary to suppose $v$ in (1) to have a value less than 1·, for the same resulting value of $w$ may be obtained by giving to $v$ its corresponding value greater than 1·. If $w$ is negative then $v$ and $\frac{1}{v}$ must be negative; and (3) becomes (4).

$$v + \frac{1}{v} = -w, \ (3);$$

$$-v - \frac{1}{v} = -w, \ (4).$$

Consequently the value of $v$ in (3) is the same as the value of $v$ in (1), numerically, but negative. And, consequently, no whole number or fraction, positive or negative, substituted for $v$ in (1) will render $v + \frac{1}{v}$ numerically less than $+2$ or $-2$. Therefore, all such equations as $v + \frac{1}{v} = \pm w$, may be put under the form (1), in which, $v$ may be always considered

greater than unity, without interfering with the continuity of the numerical values of $w$. From (1) we obtain (5),

$$1 + \frac{1}{v^2} = w\frac{1}{v}, \quad (5).$$

Since $v$ is a positive number greater than 1·, the left-hand member of (5) is always greater than 1· but less than 2·; whence the right-hand member of (5), that is, $w\,\frac{1}{v}$, must have a value existing between the same limits, but not beyond.

In supposing $w$ in (6) to have a continuous range of numerical values, no new result is obtained in supposing $v$ to be a proper fraction of the form $-\frac{1}{z}$; for then (6) becomes (7).

$$v - \frac{1}{v} = w, \quad (6);$$

$$-\frac{1}{z} + z = w, \quad (7).$$

It is therefore unnecessary to suppose $v$ in (6) to have a value less than $(-1\cdot)$; for the same resulting value of $w$ may be obtained by giving to $v$ its corresponding positive value $z$ greater than 1. Nor can $v$ in (6) be a negative whole number, for then $v - \frac{1}{v}$ would become negative; and equal to $w$ which is supposed to be positive. If $\frac{1}{z}$ be substituted for $v$ in (6) it becomes $\frac{1}{z} - z = w$, or

$$z - \frac{1}{z} = -w, \quad (8).$$

Whence, if the value of $v$ be known in (6), the value of $z$ in (8) becomes known, for $z$ in (8) is equal to $\frac{1}{v}$ in (6). $v$ cannot be $= +1$ or $-1$ in (6), for then $v - \frac{1}{v} = 0$, which is absurd, for $v$ is always $=$ the whole number $w$; $w$ being positive in (6), $v$ must be a positive whole number, for if it be a proper fraction $+\frac{1}{z}$, it assumes the form (8), in which $w$ is negative. And if $v$ be a negative fraction $-\frac{1}{z}$, (6) assumes the identical form (7), in which $z$ is a whole positive number.

Therefore all such equations as $v - \frac{1}{v} = \pm w$, may be solved by solving (6), in which $v$ may be always considered greater than unity, without interfering with the continuity of the numerical values of $+ w$.

From (6) we obtain (9),

$$1 - \frac{1}{v^2} = w\,\frac{1}{v}\,,\ (9).$$

Since $v$ in all cases is a positive number greater than 1, the left hand member of (9) is always greater than 0·, but less than 1·; whence the right-hand member of (9), that is $w\,\frac{1}{v}$, must have a value existing between the same limits 0· and 1·, but not beyond.

When $w$ is not considered as standing for all possible numbers between its known limits, but has a particular value $n$, greater than 2·; that is,

$$v + \frac{1}{v} = n;$$

$$\text{then if } a = v \text{ or } a + \frac{1}{a} = n, \text{ also will } \frac{1}{a} = v;$$

$$\text{for } \frac{1}{a} + \frac{1}{\frac{1}{a}} = \frac{1}{a} + a = n.$$

$$\text{Again in } v - \frac{1}{v} = m, \text{ then if } v = b,$$

$$\text{or } b - \frac{1}{b} = m;\ v \text{ also } = -\frac{1}{b};$$

$$\text{for, } -\frac{1}{b} - \left(\frac{1}{-\frac{1}{b}}\right) = -\frac{1}{b} + b = m.$$

---

## Lemma I.

If $\cdot f$ be put for any decimal fraction as ·34567888, and $f_1 = 3456788{\cdot}8$, $f_2 = 345678{\cdot}88$, $f_3 = 34567{\cdot}888$; &c.

Then, for dual numbers of not more than eight consecutive dual digits, we have

$$\downarrow, (1+\cdot f \downarrow u_{8,}) = \downarrow, (1+\cdot f) + \frac{f_8\, u_8}{1+\cdot f}$$

$$\downarrow, (1+\cdot f \downarrow u_{7,}) = \downarrow, (1+\cdot f) + \frac{f_7\, u_7}{1+\cdot f}$$

$$\downarrow, (1+\cdot f \downarrow u_{6,}) = \downarrow, (1+\cdot f) + \frac{f_6\, u_6}{1+\cdot f}$$

$$\downarrow, (1+\cdot f \downarrow u_{5,}) = \downarrow, (1+\cdot f) + \frac{f_5\, u_5}{1+\cdot f}$$

All of which are true to the required degree of accuracy. $f_8 = \cdot 34567888$; $f_7 = 3\cdot 4567888$; $f_6 = 34\cdot 567888$; $f_5 = 345\cdot 67888$.
$\downarrow, u_{1,} = 9531018, u_1$ ; $\downarrow, u_{2,} = 995033, u_2$ ; $\downarrow, u_{3,} = 99950, u_3$ ;
$\downarrow, u_{4,} = 10000, u_4$ ; &c.

$$\downarrow, (1+\cdot f \downarrow' u_8) = \downarrow, (1+\cdot f) - \frac{f_8\, u_8}{1+\cdot f}$$

$$\downarrow, (1+\cdot f \downarrow' u_7) = \downarrow, (1+\cdot f) - \frac{f_7\, u_7}{1+\cdot f}$$

$$\downarrow, (1+\cdot f \downarrow' u_6) = \downarrow, (1+\cdot f) - \frac{f_6\, u_6}{1+\cdot f}$$

$$\downarrow, (1+\cdot f \downarrow' u_5) = \downarrow, (1+\cdot f) - \frac{f_5\, u_5}{1+\cdot f_5}$$

Within the designed degree of accuracy, which may be as great as we please, these equations hold exactly true, while the places of figures of the whole number $f$ are not greater than half the places in $1+\cdot f$;

*Example.*

$$\begin{array}{rl} 1+\cdot 50000000 & \\ 16539 & \\ \hline 1+\cdot 50016539 & \text{Sum.} \\ 1+\cdot 49983461 & \text{Diff.} \end{array}$$

$$\downarrow, (1+\cdot 50000000) + \frac{16539}{1\cdot 5} = \downarrow, (1\cdot 5001\dot{6}539)$$

$$\downarrow, (1+\cdot 50000000) - \frac{16539}{1\cdot 5} = \downarrow, (1\cdot 49983461)$$

*Proof.*

$$\left.\begin{array}{l} \downarrow, (1\cdot 50016539) = 40557538, \\ \downarrow, (1\cdot 50000000) = 40546512, \\ \downarrow, (1\cdot 49983461) = 40535484, \end{array}\right\} \begin{array}{c} 11028, \\ \text{Differences} \\ 11028, \end{array}$$

$$\frac{16539\cdot}{1\cdot 5} = 11026.$$

Here it may be observed that, in the process hereafter established, to find the roots of equations of all degrees, the digits of the development $\downarrow u_1, u_2, u_3, \ldots$ or of $\downarrow' u_1, 'u_2, 'u_3 \ldots$ may be determined in numbers not greater than 5, or '5.

In continuation,

$\downarrow, (1+\cdot f \downarrow u_4)$ nearly equal to, but greater than $\downarrow, (1+\cdot f) + \frac{f_4 u_4}{1+\cdot f}$;

$\downarrow, (1+\cdot f \downarrow u_3)$ ,, ,, $\downarrow, (1+\cdot f) + \frac{f_3 u_3}{1+\cdot f}$;

$\downarrow, (1+\cdot f \downarrow u_2)$ ,, ,, $\downarrow, (1+\cdot f) + \frac{f_2 u_2}{1+\cdot f}$;

$\downarrow, (1+\cdot f \downarrow u_1)$ ,, ,, $\downarrow, (1+\cdot f) + \frac{f_1 u_1}{1+\cdot f}$

*Examples.*

$$\downarrow, (1+\cdot 86733389 \downarrow 0,3,) = 63849769,$$
$$\downarrow, (1+\cdot 86733389) + \frac{867333\cdot 89}{1\cdot 86733389} = 63844605.$$

Results nearly equal, but the latter is less than the former.

$$\downarrow, (1+\cdot 25000000 \downarrow 4,) = 31190496,$$
$$\downarrow, (1+\cdot 25000000) + \frac{(2500000\cdot)\times 4}{1\cdot 25} = 30299713,$$

Results approaching equality, but the latter is less than the former.

Lastly,

$\downarrow, (1+\cdot f \downarrow' u_4)$ nearly equal to, but less than $\downarrow, (1+\cdot f) - \frac{f_4 u_4}{1+\cdot f}$;

$\downarrow, (1+\cdot f \downarrow' u_3)$ ,, ,, $\downarrow, (1+\cdot f) - \frac{f_3 u_3}{1+\cdot f}$;

$\downarrow, (1+\cdot f \downarrow' u_2)$ ,, ,, $\downarrow, (1+\cdot f) - \frac{f_2 u_2}{1+\cdot f}$;

$\downarrow, (1+\cdot f \downarrow' u_1)$ ,, ,, $\downarrow, (1+\cdot f) - \frac{f_1 u_1}{1+\cdot f}$

*Examples.*

$$\downarrow, (1+\cdot 86733389 \downarrow' 0'3) = 61062079,$$
$$\downarrow, (1+\cdot 86733389) - \frac{(867333\cdot 89)\times 3}{1\cdot 86733389} = 61057741.$$

These results are also nearly equal; the latter, as in the former case, being less than the former.

$$\downarrow, (1+\cdot25000000\downarrow'4) \qquad = 15188386,$$
$$\downarrow, (1+\cdot25000000) - \frac{(2500000\cdot)\times 4}{1\cdot25} = 14299713.$$

Results approaching equality, but the latter is less than the former as in previous cases.

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## Lemma II.

As in Lemma I., if $\cdot f$ be put for any decimal fraction—$\cdot34567822$ and $f_1{=}3456782\cdot2$; $f_2{=}345678\cdot22$; $f_3{=}34567\cdot822$; &c.

Then, for dual numbers of not more than eight consecutive dual digits, we have—

$$\downarrow, (1-\cdot f\downarrow u_{8,}) = \downarrow, (1-\cdot f) + \frac{f_8 u_8}{1-\cdot f};$$
$$\downarrow, (1-\cdot f\downarrow u_{7,}) = \downarrow, (1-\cdot f) + \frac{f_7 u_7}{1-\cdot f};$$
$$\downarrow, (1-\cdot f\downarrow u_{6,}) = \downarrow, (1-\cdot f) + \frac{f_6 u_6}{1-\cdot f};$$
$$\downarrow, (1-\cdot f\downarrow u_{5,}) = \downarrow, (1-\cdot f) + \frac{f_5 u_5}{1-\cdot f}.$$

These equations coincide exactly, while the places of figures in the whole number $f\cdot$ are not greater than half the places of figures in $1-\cdot f$.

*Examples.*

| | | |
|---|---|---|
| $1-\cdot18000000$ | $=\cdot82000000$ | |
| $1-\cdot18009237$ | $=\cdot81990763$ | |
| 9237 | 9237 | difference, |

which, when taken as a whole number, gives

$$\frac{9237\cdot}{\cdot82} = '11267, \text{ the difference of the dual}$$

logarithms of $\cdot82000000$ and $\cdot81990763$.

*Proof.*

$$\downarrow, (1 - \cdot18009237) = '19856360 = \downarrow, (\cdot81990763)$$
$$\downarrow, (1 - \cdot18000000) = '19845093 = \downarrow, (\cdot82000000)$$
$$'11267 \text{ difference.}$$

*Ex.* 2.

$$\downarrow, (1 - \cdot 27702856) \qquad\qquad = {}'32438457$$

$$\downarrow, (1 - \cdot 27700000) + \frac{2856\cdot}{\cdot 723} = {}'32438457$$

$$\text{For } \downarrow, (1 - \cdot 2700000) = {}'32434506 = \downarrow, (\cdot 723)$$

$$\text{and } \frac{2856}{\cdot 723} = \frac{'3951}{'32438457}$$

When the digits are of the descending branch then $\downarrow, {}'u_1 = {}'10536052 u_1$ ; $\downarrow, {}'u_2 = {}'1005034 u_2$ ; $\downarrow, {}'u_3 = {}'100050 u_3$ ; $\downarrow, {}'u_4 = {}'10001 u_4$ ; &c.

$$\downarrow, (1 - \cdot f \downarrow {}'u_8) = \downarrow, (1 - \cdot f) - \frac{f_8 u_8}{1 - \cdot f};$$

$$\downarrow, (1 - \cdot f \downarrow {}'u_7) = \downarrow, (1 - \cdot f) - \frac{f_7 u_7}{1 - \cdot f};$$

$$\downarrow, (1 - \cdot f \downarrow {}'u_6) = \downarrow, (1 - \cdot f) - \frac{f_6 u_6}{1 - \cdot f};$$

$$\downarrow, (1 - \cdot f \downarrow {}'u_5) = \downarrow, (1 - \cdot f) - \frac{f_5 u_5}{1 - \cdot f}.$$

If $1 - \cdot f = 1 - \cdot 25000000$; then $f_8 = \cdot 25000000$; $f_7 = 2\cdot 5000000$; $f_6 = 25\cdot 000000$; $f_5 = 250\cdot 00000$; $f_4 = 2500\cdot 0000$; &c.

*Example.*

$$\downarrow, (1 - \cdot 30000000 \downarrow {}'7_5) \qquad\qquad = {}'35664495$$

$$\downarrow, (1 - \cdot 30000000) - \frac{(300\cdot) \times 7}{1 - \cdot 3} = {}'35664495$$

$$\text{In this example } f_5 = 300\cdot 00000$$

$$\frac{(300\cdot) \times (7)}{1 - \cdot 3} = {}'3000$$

$\downarrow, (\cdot 70000000) = {}'35667495$ ; $\downarrow, (1 - \cdot 29997900) = {}'35664495$.

Generally $\downarrow, (1 - \cdot f \downarrow {}'u_\nu) = \downarrow, (1 - \cdot f) \pm \frac{f_n \times u_n}{1 - \cdot f}$, holds true to the designed degree of accuracy, which may be as great as we please, while the places of figures in the whole number represented by $f_n \times u_n$ are not greater than half the places of figures in $1 - \cdot f$.

$\downarrow, (1 - \cdot f \downarrow u_4,)$ nearly =, but greater than $\downarrow, (1 - \cdot f) + \frac{f_4 u_4}{1 - \cdot f}$;

$\downarrow, (1 - \cdot f \downarrow u_3,)$ " " $\downarrow, (1 - \cdot f) + \frac{f_3 u_3}{1 - \cdot f}$;

$\downarrow, (1 - \cdot f \downarrow u_{2,}) $ nearly $=$, but greater than $\downarrow, (1 - \cdot f) + \frac{f_2 u_2}{1 - \cdot f}$;

$\downarrow, (1 - \cdot f \downarrow u_{1,})$ „ „ $\downarrow, (1 - \cdot f) + \frac{f_1 u_1}{1 - \cdot f}$;

*Examples.*

$$\text{Let } 1 - \cdot f \downarrow u_{2,} = 1 - \cdot 28000000 \downarrow 0{,}2,$$

$$\text{then } u_{2,} = 2 \text{ and } f\cdot = 280000.$$

$$\downarrow, (1 - \cdot 28000000 \downarrow 2_{2,}) = \text{'}33635146$$

$$\downarrow, (1 - \cdot 28000000) + \frac{(280000.) \times 2}{1 - \cdot 28} = \text{'}33628185$$

These results approach equality, but the latter is less than the former.

$$\downarrow, (1 - \cdot 28000000) = \downarrow, (\cdot 72000000) = \text{'}32850408$$

$$\text{and } \frac{(280000\cdot) \times 2}{\cdot 72} = \text{'}777777.$$

*Ex.* 2.

Let $1 - \cdot f \downarrow u_1 = 1 - \cdot 20000000 \downarrow 3$, then $u_1 = 3$ and $f\cdot = 2000000$.

$$\downarrow, (1 - \cdot 20000000 \downarrow 3_{,}) = \text{'}30951877$$

$$\downarrow, (1 - \cdot 20000000) + \frac{(2000000\cdot) \times 3}{1 - \cdot 2} = \text{'}29689437.$$

These results also approach equality, but the latter is less than the former.

$$\downarrow, (1 - \cdot 20000000) = \downarrow, (\cdot 80000000) = \text{'}22189437$$

$$\text{and } \frac{(2000000\cdot) \times 3}{1 - \cdot 2} = 7500000.$$

Lastly,

$\downarrow, (1 - \cdot f \downarrow \text{'} u_4)$ nearly $=$, but greater than $\downarrow, (1 - \cdot f) - \frac{f_4 u_4}{1 - \cdot f}$;

$\downarrow, (1 - \cdot f \downarrow \text{'} u_3)$ „ „ $\downarrow, (1 - \cdot f) - \frac{f_3 u_3}{1 - \cdot f}$;

$\downarrow, (1 - \cdot f \downarrow \text{'} u_2)$ „ „ $\downarrow, (1 - \cdot f) - \frac{f_2 u_2}{1 - \cdot f}$;

$\downarrow, (1 - \cdot f \downarrow \text{'} u_1)$ „ „ $\downarrow, (1 - \cdot f) - \frac{f_1 u_1}{1 - \cdot f}$.

*Ex.* 1.

$$\downarrow, (1 - \cdot 28000000 \downarrow \text{'} 2_2) = \text{'}32079496$$

$$\downarrow, (1 - \cdot 28000000) - \frac{(280000\cdot) \times 2}{1 - \cdot 28} = \text{'}32072631$$

*Ex.* 2.

$$\downarrow, (1 - \cdot 20000000 \downarrow '3) \qquad = '15758995$$
$$\downarrow, (1 - \cdot 20000000) \quad - \quad \frac{(2000000\cdot) \times 3}{1 - \cdot 2} \qquad = '14689437$$

Generally, if $\downarrow u_1 u_2 u_3 \ldots\ldots u_n$ be a dual development, and $u_1 u_2 u_3 \ldots\ldots$ either ascending or descending dual digits; then if $p$ be greater than half $n$, and $q$ not greater than half $n$,

$$\downarrow, (1 + \cdot f \downarrow u_p) = \downarrow, (1 + \cdot f) + \frac{f_p u_p}{1 + \cdot f};$$

$$\text{and } \downarrow, (1 + \cdot f \downarrow ' u_p) = \downarrow, (1 + \cdot f) - \frac{f_p u_p}{1 + \cdot f};$$

true to the designed degree of accuracy, which may be as great as we please. But,

$\downarrow, (1 + \cdot f \downarrow u_q,)$ is nearly $=$, but greater than $\downarrow, (1 + \cdot f) + \frac{f_q u_q}{1 + \cdot f};$

and

$\downarrow, (1 + \cdot f \downarrow ' u_q)$ „ „ $\downarrow, (4 + \cdot f) - \frac{f_q u_q}{1 + \cdot f};$

Also, generally,

$$\downarrow, (1 - \cdot f \downarrow u_p) = \downarrow, (1 - \cdot f) + \frac{f_p u_p}{1 - \cdot f};$$

$$\downarrow, (1 - \cdot f \downarrow ' u_p) = \downarrow, (1 - \cdot f) - \frac{f_p u_p}{1 - \cdot f};$$

true to the designed degree of accuracy; but

$\downarrow, (1 - \cdot f \downarrow u_q,)$ is nearly $=$, but greater than $\downarrow, (1 - \cdot f) + \frac{f_q u_q}{1 - \cdot f};$

and

$\downarrow, (1 - \cdot f \downarrow ' u_q)$ „ „ $\downarrow, (1 - \cdot f) - \frac{f_q u_q}{1 - \cdot f};$

When operating with the digits $'u_q\ u_q,$ it will be found that, in these latter cases, the inequalities are of much importance, when results are sought in their lowest terms.

To solve any equation of the form

$$1 + \frac{1}{y^2} = c \frac{1}{y^m}; \qquad \text{(I).}$$

In which $m$ represents any given number, whole or fractional, positive or negative; the coefficient $c$ being also given.

$$\therefore \downarrow, \left(1+\frac{1}{y^2}\right) = \downarrow, (c) - m \downarrow, (y); \qquad \text{(II)}.$$

We have before shown that all numerical values of $1+\frac{1}{y^2}$ exist between the limits 1 and 2; hence, by putting

$$\downarrow, (c) - m\downarrow, (y) = \downarrow, (2)$$

$$\downarrow, (y) = \frac{\downarrow, (c) - \downarrow, (2)}{m}; \qquad \text{(III)}.$$

Again, if $\downarrow, (c) - m\downarrow, (y)$ be put $= 0$, then

$$\downarrow, (y) = \frac{\downarrow, (c)}{m}, \qquad \text{(IV)};$$

from which the other limiting value of $y$ is obtained.

Now suppose L to be any convenient number existing between these limits, in putting $\frac{1}{s} = \frac{1}{L^2}$ and substituting $\frac{1}{s}\downarrow u_1$, or $\frac{1}{s}\uparrow' u_1$ for $\frac{1}{y^2}$ in (II) we have in the first case

$$\downarrow, \left(1+\frac{1}{s}\downarrow u_1,\right) = \downarrow, (c) - m\downarrow, \left(s^{\frac{1}{2}} \frac{1}{\downarrow \frac{u_1}{2},}\right)$$

$$\therefore \downarrow, \left(1+\frac{1}{s}\right) + \frac{\frac{1}{s} u_1}{1+\frac{1}{s}} = \downarrow, (c) - \frac{m}{2} \downarrow, (s) + \frac{m}{2} \downarrow, u_1,$$

$$\therefore u_1, = \frac{\downarrow, (c) - \frac{m}{2}\downarrow, (s.) - \downarrow, \left(1+\cdot\frac{1}{s}\right)}{\frac{\frac{1}{s}}{1+\cdot\frac{1}{s}} - \frac{m}{2}\ (9531018,)}$$

$$\text{or } u_1, = \frac{\downarrow, (c) - \frac{m}{2} \downarrow, \left(\frac{1}{f_1}\right) - \downarrow, (1+\cdot f_1)}{\frac{f_1}{1+\cdot f_1} - \frac{m}{2}\ (9531018,)}, \qquad \text{(V)};$$

according to the notation previously established.

$$u_2, = \frac{\downarrow, (c) - \frac{m}{2} \downarrow, \left(\frac{1}{\cdot f_2}\right) - \downarrow, (1+\cdot f_2)}{\frac{f_2}{1+\cdot f_2} - \frac{m}{2}\ (995033,)}, \qquad \text{(VI)};$$

In which $\cdot f_2 = f_1\downarrow u_1, = \frac{1}{s}\downarrow u_1,$

$$u_{3}, = \frac{\downarrow,(c) - \frac{m}{2}\downarrow,\left(\frac{1}{\cdot f_3}\right) - \downarrow,(1+\cdot f_3)}{\frac{f_3}{1+\cdot f_3} - \frac{m}{2}\ (99950,)}, \quad \text{(VII)};$$

In which $\cdot f_3 = \cdot f_2 \downarrow u_2, = \cdot f_1 \downarrow u_1, u_2, = \frac{1}{s}\downarrow u_1, u_2,$;

$$u_{4}, = \frac{\downarrow,(c) - \frac{m}{2}\downarrow,\left(\frac{1}{\cdot f_4}\right) - \downarrow,(1+\cdot f_4)}{\frac{f_4}{1+\cdot f_4} - \frac{m}{2}\ (10000,)}, \quad \text{(VIII)};$$

In which $\cdot f_4 = \cdot f_3 \downarrow u_3, = \cdot f_2 \downarrow u_2, u_3, = \cdot f_1, \downarrow u_1, u_2, u_3.$ This process may be continued to any required extent, and to any degree of accuracy.

When $\frac{1}{y_2} = \frac{1}{s}\downarrow u_1, u_2, u_3, \ldots\ldots$ becomes known, $y$ is easily found. In order to find a dual expression for $\frac{1}{y_2}$ in its lowest terms, it is important that we are able to select either a descending or an ascending dual digit at any stage of the development.

In the second case, if $\frac{1}{s}\downarrow{}'u_1$ be substituted for $\frac{1'}{y_2}$ in (II), we have

$$\downarrow,\left(1+\frac{1}{s}\downarrow{}'u_1\right) = \downarrow,(c) - m\downarrow,\left(s^{\frac{1}{2}}\frac{1}{\downarrow\frac{'u_1}{2}}\right)$$

$$\therefore \downarrow,\left(1+\frac{1}{s}\right) - \frac{\frac{1}{s}u_1}{1+\frac{1}{s}} = \downarrow,(c) - \frac{m}{2}\downarrow,(s) + \frac{m}{2}\downarrow,{}'u_1$$

$$\therefore {}'u_1 = \frac{\downarrow,(c) - \frac{m}{2}\downarrow,(s) - \downarrow,\left(1+\frac{1}{s}\right)}{-\frac{\frac{1}{s}}{1+\frac{1}{s}} + \frac{m}{2}\ (10536052)}$$

$$\text{or } {}'u_1, = \frac{\downarrow,(c) - \frac{m}{2}\downarrow,\left(\frac{1}{f_1}\right) - \downarrow,(1+\cdot f_1)}{-\frac{f_1}{1+\cdot f_1} + \frac{m}{2}\ (10536052)}; \quad \text{(IX)},$$

according to the established notation.

$$'u_2 = \frac{\downarrow,(c) - \frac{m}{2}\downarrow,\left(\frac{1}{\cdot f_2}\right) - \downarrow,(1+\cdot f_2)}{-\frac{f_2}{1+\cdot f_2} + \frac{m}{2}(1005034)}; \quad (X);$$

In which $\cdot f_2 = \cdot f_1 \downarrow 'u, = \frac{1}{8} \downarrow 'u_1$;

$$'u_3 = \frac{\downarrow,(c) - \frac{m}{2}\downarrow,\left(\frac{1}{\cdot f_3}\right) - \downarrow,(1+\cdot f_3)}{-\frac{f_3}{1+\cdot f_3} + \frac{m}{2}(100050)}; \quad (XI);$$

In which $\cdot f_3 = \cdot f_2 \downarrow 'u_2 = f_1, \downarrow 'u_1, 'u_2 = \frac{1}{8} \downarrow 'u_1, 'u_2$

$$'u_4 = \frac{\downarrow,(c) - \frac{m}{2}\downarrow,\left(\frac{1}{\cdot f_4}\right) - \downarrow,(1+\cdot f_4)}{-\frac{f_4}{1+\cdot f_4} + \frac{m}{2}(10000)}; \quad (XII);$$

In which $\cdot f_4 = \cdot f_3 \downarrow 'u_3 = \cdot f_2 \downarrow 'u_2\, 'u_3 = \cdot f_1, \downarrow 'u_1, 'u_2\, 'u_3$

$\therefore$ the value of $\frac{1}{y^2}$ may be found under the dual form $\frac{1}{8} \downarrow u_1, u_2\, u_3\, u_4 \ldots$ to any required degree of accuracy, and this form may be found in its lowest terms, by a direct process, without resorting to the different systems of trial and error practised in applying all other known methods for finding the roots of equations.

A general Quintic, or equation of the fifth degree of the form

$$x^5 + Ax + Bx^3 + Cx^2 + Dx + E = 0$$

may be deprived of any three of its four centre terms by solving equations of its inferior degrees, and reduced to any of the following forms:—

$$x^5 + ax = e;$$
$$x^5 + bx^2 = e;$$
$$x^5 + cx^3 = e;$$
$$x^5 + dx^4 = e;$$

These are readily put under the soluble form (I).

### *Ex.* 1.

Solve the general equation $x^5 + ax = e$; and find the value of $x$ in the equation

$$x^5 + 625x = 25000.$$

$$x^5 + ax = e \text{ or } x^2 + \frac{a}{x^2} = \frac{e}{x^3}$$

$$\text{Put } x^2 = ya^{\frac{1}{2}} \text{ then } x^3 = a^{\frac{3}{4}}\ y^{\frac{3}{2}}$$

$$\therefore x^2 + \frac{a}{x^2} = \frac{e}{x^3} \text{ becomes } ya^{\frac{1}{2}} + \frac{a}{ya^{\frac{1}{2}}} = \frac{e}{a^{\frac{3}{4}}}\ \frac{1}{y^{\frac{3}{2}}}$$

$$\therefore y + \frac{1}{y} = \frac{e}{a^{\frac{5}{4}}}\ \frac{1}{y^{\frac{3}{2}}}$$

$$\therefore 1 + \frac{1}{y} = \frac{e}{a^{\frac{5}{4}}}\ \frac{1}{y^{\frac{5}{2}}}$$

Hence, comparing the particular example above given with general form (I), we have $\frac{e}{a^{\frac{5}{4}}} = c$, and $m = \frac{5}{2}$.

$$\downarrow, (c) = 207944154,$$

$$\therefore \text{From (II), } \downarrow, (1 + \frac{1}{y^2}) = 207944154, - \tfrac{5}{2} \downarrow, (y)$$

To find the limiting values of $y$;

$$\text{From (III), } \downarrow, (y) = \frac{\downarrow, (c) - \downarrow, (2)}{m} = 55451770,$$

$$\therefore y = 1{\cdot}74110870.$$

$$\text{From (IV), } \downarrow, (y) = \frac{\downarrow, (c)}{m} = 83177662,$$

$$\therefore y = 2{\cdot}29739673$$

Consequently, $y$ must have a value between 1·74110870 and 2·29739673.

Then L may be put = 2· and $\frac{1}{L^2} = \frac{1}{4} = {\cdot}25000000 = \frac{1}{s} = {\cdot}f_1$

$$\therefore 1 + {\cdot}f_1 = 1{\cdot}25000000, \text{ and } f_1 = 2500000.$$

Hence, from (IX) we obtain

$$'u, = \frac{\downarrow, (c) - \frac{m}{2} \downarrow, \left(\frac{1}{{\cdot}f}\right) - \downarrow, (1 + {\cdot}f_1)}{-\frac{f_1}{1 + {\cdot}f_1} + \frac{m}{2}(10536052)} = \frac{12343004}{11170065} = \downarrow\ '1$$

$$\begin{array}{rl} \text{Then,} & {\cdot}25000000 \\ & 2500000 \\ \hline & {\cdot}22500000 = {\cdot}f_2 \\ & 225000. = f_2 \end{array}$$

From (X) we find,

$$'u_2 = \frac{\downarrow,(c) - \frac{m}{2}\downarrow,\left(\frac{1}{\cdot f_2}\right) - \downarrow,(1+\cdot f_2)}{-\frac{f_2}{1+\cdot f_2} + \frac{m}{2}(1005034)} = \frac{1193202}{1072620} = \downarrow '0\,'1$$

$$\begin{array}{rl} \text{Again,} & \cdot 22500000 \\ & 225000 \\ \hline & \cdot 22275000 = \cdot f_3 \\ & 22275. = f_3 \end{array}$$

Then from (XI) we find,

$$'u_3 = \frac{\downarrow,(c) - \frac{m}{2}\downarrow\left(\frac{1}{\cdot f_3}\right) - \downarrow,(1+\cdot f_3)}{-\frac{f_3}{1+\cdot f_3} + \frac{m}{2}(100050)} = \frac{120758}{106845} = \downarrow '0\,'0\,'1_3\,'1_4\,'3_5$$

$$\begin{array}{rl} \text{Lastly,} & \cdot 22275\,0000 \\ & 22\,2750 \\ \hline & \cdot 22252\,7250 \\ & 2\,2253 \\ \hline & \cdot 22250|4997 \\ & |6675 \\ \hline & \cdot 22249\,8322 \\ \therefore & \cdot 22249\,8322 = \cdot f_6 \\ & 22. = f_6 \end{array}$$

$$\therefore\ 'u_6 = \frac{\downarrow,(c) - \frac{m}{2}\downarrow,\left(\frac{1}{\cdot f_6}\right) - \downarrow,(1+\cdot f_6)}{-\frac{f_6}{1+\cdot f_6} + \frac{m}{2}(100)} = \downarrow '0_1\,'0_2\,'0_3\,'0_4\,'0_5\,'0_6\,'3_7\,'0_8 = \frac{30}{107}.$$

$$\therefore \frac{1}{y^2} = \frac{1}{4}\downarrow '1\,'1\,'1\,'1\,'3\,'0\,'3\,'0$$

$$\therefore \downarrow,(y) = 75141801,$$

$$x = y^{\frac{1}{2}}\,a^{\frac{1}{4}} \text{ and } \downarrow,(x) = \tfrac{1}{2}\downarrow,(y) + \tfrac{1}{4}\downarrow,(a),$$

$$\therefore \downarrow,(x) = 198514691,$$

$$\therefore x = 7{\cdot}28011692.$$

When a quintic equation is reduced to the form

$$x^5 + bx^3 = e\,;$$

divide by $x^{\frac{7}{2}}$; $\frac{7}{2} = 5 + 2$ divided by 2.

$$\text{then } x^{\frac{3}{2}} + \frac{b}{x^{\frac{1}{2}}} = \frac{e}{x^{\frac{7}{2}}}.$$

Putting $x^{\frac{3}{2}} = yb^{\frac{1}{2}}$, hence $x^{\frac{7}{2}} = y^{\frac{7}{3}}\, b^{\frac{7}{6}}$

$$\therefore yb^{\frac{1}{2}} + \frac{b}{yb^{\frac{1}{2}}} = \frac{e}{y^{\frac{7}{3}}\, b^{\frac{7}{6}}} \text{ or } y + \frac{1}{y} = \frac{e}{y^{\frac{7}{3}}\, b^{\frac{2}{3}}}$$

$$\therefore 1 + \frac{1}{y^2} = \frac{e}{b^{\frac{2}{3}}}\ \frac{1}{y^{\frac{10}{3}}}$$

$$\therefore \downarrow, (1 + \frac{1}{y^2}) = \downarrow, \left(\frac{e}{b^{\frac{2}{3}}}\right) - \tfrac{10}{3} \downarrow, (y), \text{ which}$$

corresponds with the general form (II), and may be solved in a similar manner.

Any equation of the form $x^m + bx^n = e$, may be reduced to (II) by making similar substitutions, and then solved by the general method.

$$\text{Dividing by } x^{\frac{m+n}{2}} \text{ we have}$$

$$x^{\frac{m-n}{2}} + \frac{b}{x^{\frac{m-n}{2}}} = \frac{e}{x^{\frac{m+n}{2}}}, \text{ in which}$$

$$\text{put } x^{\frac{m-n}{2}} = yb^{\frac{1}{2}}; \text{ then we obtain}$$

$$y + \frac{1}{y} = \frac{e}{b^{\frac{m}{m-n}}}\ \frac{1}{y^{\frac{m+n}{m-n}}}$$

$$\therefore 1 + \frac{1}{y^2} = \frac{e}{b^{\frac{m}{m-n}}}\ \frac{1}{y^{\frac{2m}{m-n}}}$$

$$\therefore \downarrow, (1 + \frac{1}{y^2}) = \downarrow, \left(\frac{e}{b^{\frac{m}{m-n}}}\right) - \frac{2m}{m-n} \downarrow, (y); \quad \text{(A).}$$

Putting, as before, $c$ for the known quantity $\frac{e}{b^{\frac{m}{m-n}}}$; $'f$, for the fraction $\frac{1}{s}$ reduced to a decimal; $f_1$; $f_2$; $f_3$; &c., for the whole numbers before defined; and $k_1$, for 9531018,; $k_2$, for 995033,; $k_3$, for 99950,; &c. $'k_1$ for '10536052; $'k_2$ for '1005034; $'k_3$ for '100050; &c. Without affecting the general formulæ, numerical values may be given to $k_1$ $k_2$ $k_3$ . . . . to suit any required degree of accuracy; in the present case, the range of accuracy consists of eight consecutive dual digits. Having

determined the limits $L_1$; $L_2$; of (A), take L = any number existing between these limits and assume

$$\frac{1}{s} = \frac{1}{L^2} \text{ and } \frac{1}{s} \downarrow u_1, = \frac{1}{y^2}$$

$$\therefore \downarrow,\left(\frac{1}{s} \downarrow u_1,\right) = \downarrow,\left(\frac{1}{y^2}\right)$$

$$\therefore -\frac{1}{2}\downarrow,(s) + \frac{1}{2}\downarrow,u_1, = -\downarrow,(y)$$

Consequently (A) becomes

$$\downarrow,\left(1+\frac{1}{s}\downarrow u_1,\right) = \downarrow,(c) - \frac{m}{m-n}\downarrow,(s) + \frac{m}{m-n}\downarrow,u_1,$$

$$\therefore \downarrow,\left(1+\frac{1}{s}\right) + \frac{\left(\frac{1}{s}\right).\times u_1}{1+\frac{1}{s}} = \downarrow,(c) - \frac{m}{m-n}\downarrow,(s) + \frac{m}{m-n}(k_1) \times u_1$$

$$\therefore u_1 = \frac{\downarrow,(c) - \frac{m}{m-n}\downarrow,(s) - \downarrow,\left(1+\frac{1}{s}\right)}{\frac{\left(\frac{1}{s}\right).}{1+\frac{1}{s}} - \frac{m}{m-n}(k_1)} = \frac{\downarrow,(c) - \frac{m}{m-n}\downarrow\left(\frac{1}{\cdot f}\right) - \downarrow(1+\cdot f)}{\frac{f_1}{1+\cdot f} - \frac{m}{m-n}(k_1)};$$

$$\text{or } u_1, = \frac{\frac{m}{m-n}\downarrow,\left(\frac{1}{\cdot f}\right) + \downarrow,(1+\cdot f) - \downarrow,(c)}{\frac{m}{m-n}(k_1) - \frac{f_1}{1+\cdot f}}$$

$$\text{Generally, } u_p, = \frac{\frac{m}{m-n}\downarrow,\left(\frac{1}{\cdot f}\right) + \downarrow,(1+\cdot f) - \downarrow,(c)}{\frac{m}{m-n}(k_p) - \frac{f_p}{1+\cdot f}}; \quad \text{(B).}$$

Again, if $\frac{1}{s}\downarrow{}'u_1$ be put $= \frac{1}{y^2}$

$$-\frac{1}{2}\downarrow,(s) + \frac{1}{2}\downarrow,{}'u_1 = -\downarrow,(y), \text{ and (A) becomes}$$

$$\downarrow,\left(1+\frac{1}{s}\right) - \frac{\left(\frac{1}{s}\right)\times u_1}{1+\frac{1}{s}} = \downarrow,(c) - \frac{m}{m-n}\downarrow,(s) + \frac{m}{m-n}\downarrow,{}'u_1$$

$$\therefore u_1 = \frac{\downarrow,(c) - \frac{m}{m-n}\downarrow,(s) - \downarrow,\left(1+\frac{1}{s}\right)}{\frac{-\frac{1}{s}}{1+\frac{1}{s}} + \frac{m}{m-n}(k_1)} = \frac{\downarrow,(c) - \frac{m}{m-n}\downarrow,\left(\frac{1}{\cdot f}\right) - \downarrow,(1+\cdot f)}{-\frac{f_1}{1+\cdot f} + \frac{m}{m-n}(k_1)}$$

$$\text{Or } {}'u_1, = \frac{\downarrow,(c) - \frac{m}{m-n}\downarrow\left(\frac{1}{\cdot f}\right) - \downarrow,(1+\cdot f)}{\frac{m}{m-n}(k_1) - \frac{f_1}{1+\cdot f}};$$

$$\text{Since } + \frac{m}{m-n}\downarrow,{}'u_1 = -\frac{m}{m-n}(k_1)\,u_1$$

$$\text{Generally, } {}'u_p = \frac{\downarrow,(c) - \frac{m}{m-n}\downarrow,\left(\frac{1}{\cdot f}\right) - \downarrow,(1+\cdot f)}{\frac{m}{m-n}(k_p) - \frac{f_p}{1+\cdot f}}; \text{ (C).}$$

When (A) is of the form $\downarrow,\left(1 - \frac{1}{y_2}\right) = \downarrow,(c) - \frac{2m}{m-n}\downarrow,(y)$, then (B) becomes,

$$u_p, = \frac{\frac{m}{m-n}\downarrow,\left(\frac{1}{\cdot f}\right) + \downarrow,(1-\cdot f) - \downarrow,(c)}{\frac{m}{m-n}(k_p) - \frac{f_p}{1-\cdot f}}; \text{ (D).}$$

But when (A) is of the form $\downarrow,\left(1 - \frac{1}{y^2}\right) = \downarrow,(c) - \frac{2m}{m-n}\downarrow,(y)$, and $\frac{1}{y^2}$ put $= \frac{1}{s}\downarrow{}'u_p$ then (C) becomes

$${}'u_p = \frac{\downarrow,(c) - \frac{m}{m-n}\downarrow,\left(\frac{1}{\cdot f}\right) - \downarrow,(1-\cdot f)}{\frac{m}{m-n}(k_p) - \frac{f_p}{1-\cdot f}}; \qquad \text{(E).}$$

In the expression $y + \frac{1}{y} = c\frac{1}{y^{\frac{m+n}{m-n}}}$ when $c$ is less than 1, it is evident that $y$ must be less than 1, also.

Then put $y = \frac{1}{z}$ and the expression becomes

$$z + \frac{1}{z} = c\, z^{\frac{m+n}{m-n}};$$

$\therefore 1 + \frac{1}{z^2} = c\, z^{\frac{2n}{m-n}}$ and $\downarrow,(1 + \frac{1}{z^2}) = \downarrow,(c) + \frac{2n}{m-n}\downarrow,(z)$

If $\frac{1}{s}\downarrow{}'u_1$ be substituted for $\frac{1}{z^2}$

$$-\downarrow,(s) + \downarrow,{}'u_1 = -2\downarrow,(z)$$

$$\therefore \frac{1}{2}\downarrow,(s) - \frac{1}{2}\downarrow,{}'u_1 = \downarrow,(z)$$

and $\downarrow,(1 + \frac{1}{s}\downarrow,{}'u_1) = \downarrow,(c) + \frac{n}{m-n}\downarrow,(s) - \frac{n}{m-n}\downarrow,{}'u_1$

$$\therefore \downarrow, \left(1+\frac{1}{s}\right) - \frac{\left(\frac{1}{s}\right)\times u_1}{1+\frac{1}{s_\prime}} = \downarrow, (c) + \frac{n}{m-n}\downarrow, (s) - \frac{n}{m-n}\,('k_1)\, u_1$$

$$\therefore {}'u_1 = \frac{\downarrow, (c) + \frac{n}{m-n}\downarrow, \left(\frac{1}{\cdot f}\right) - \downarrow, (1+\cdot f)}{-\frac{n}{m-n}k_1 - \frac{f_1}{1+\cdot f}} =$$

$$\frac{\downarrow, (1+\cdot f) - \frac{n}{m-n}\downarrow, \left(\frac{1}{\cdot f}\right) - \downarrow, (c)}{\frac{n}{m-n}k_1 + \frac{f_1}{1+\cdot f}}$$

$$\therefore {}'u_p = \frac{\downarrow, (1+\cdot f) - \frac{n}{m-n}\downarrow, \left(\frac{1}{\cdot f}\right) - \downarrow, (c)}{\frac{n}{m-n}k_p + \frac{f_p}{1+\cdot f}}\;;\;(G).$$

By reasoning in a similar manner we find that

$$u_{p}, = \frac{\downarrow, (c) + \frac{n}{m-n}\downarrow, \left(\frac{1}{\cdot f}\right) - \downarrow, (1+\cdot f)}{\frac{f_p}{1+\cdot f} + \frac{n}{m-n}(k_p)}\;;\;(F).$$

*Ex.* 2.

Given $x^{291} + 500\, x^{166} = 870000\cdot$ to find $x$.

$$\downarrow, \left(\frac{e}{b^{\frac{m}{m-n}}}\right) = {}'79135917 = \downarrow, (c).$$

This result being a logarithm of the descending branch, formulæ (F) and (G) have to be applied

*To find the limits.*

Put $\downarrow, (c) + \frac{2n}{m-n}\downarrow, (z) = \downarrow, (2)$, then

$\downarrow, (z) = 55892556$, and $z = 1\cdot74879245$

If $\downarrow, (c) + \frac{2n}{m-n}\downarrow, (z)$ be put $= 0$, we have

$\downarrow, (z) = 29795149$, and $z = 1\cdot34709644$

Then $z$ may be taken $=$ any value between these limits.

In (F) putting $1 + \cdot 40000000 \downarrow u, = 1 + \frac{1}{y^2}$;

$\therefore 1 + \cdot f = 1 \cdot 40000000$, and $\cdot f = \cdot 40000000$;

then $\downarrow, (c) + \frac{n}{m-n} \downarrow, \left(\frac{1}{\cdot f}\right) - \downarrow, (1 + \cdot f) = 8900264$,

which indicates that $p$ must be $= 2$.

$\because \frac{f_2}{1+\cdot f} = \frac{400000\cdot}{1\cdot 4} = 285714\cdot$; $\frac{n}{m-n} k_2 = \frac{166}{125} (995033,) = 1321403$

and $\frac{8900264}{1607118} = 5, 5 = u_2\ u_3$.

$$
\begin{array}{r}
\cdot 40|00|00|00|0 \\
2|00|00|00|0 \\
4|00|00|0 \\
4|00|0 \\
2|0 \\
\hline
\cdot 420|4\,04|02\,0 \\
2|1\,02|02\,0 \\
4|20\,4 \\
4 \\
\hline
\cdot 422\,5\,10\,24\,8 = \cdot f
\end{array}
$$

Again, putting $1 + \cdot 42251025 \downarrow u$, for $1 + \frac{1}{y^2}$, we have

$\downarrow, (c) + \frac{n}{m-n} \downarrow, \left(\frac{1}{\cdot f}\right) - \downarrow, (1 + \cdot f) = 34490$,

which shows that $p$ must be taken $= 4$.

$\therefore \frac{f_4}{1+\cdot f} = \frac{4225}{1\cdot 4225} = 2970\cdot$ and $\frac{n}{m-n} k_4 = 13280\cdot$

$$
\begin{array}{l}
\;13280\cdot \\
\;\;\;2970\cdot \\
\hline
\;16250\cdot)\ 34490\cdot\ (2, 1, 2, 2, 6,
\end{array}
$$

$\therefore \cdot 4 \downarrow, 0, 5, 5, 2, 1, 2, 2, 6, = \frac{1}{z^2}$

$\therefore \downarrow, (z) = 43066465$,

But $\downarrow, (x) = \dfrac{2 \downarrow, \left(\frac{1}{z}\right) + \downarrow, (b)}{m-n} = 4282623$, $\therefore x = 1\cdot 0437565$.

### *Ex.* 3.

Find the roots of the equation $x^2 - ax + b = 0$, and apply the general formulæ to the particular case $x^2 - 1866{\cdot}58714x + 649539$.

See 'Dual Arithmetic, a new Art,' Part II. p. 168.

Dividing by $x$ we have $x + \frac{b}{x} = a$, then putting $x = yb^{\frac{1}{2}}$, we obtain $y + \frac{1}{y} = \frac{a}{b^{\frac{1}{2}}}$. Since $\frac{a}{b^{\frac{1}{2}}}$, in the present case, is greater than 2, the equation has two real roots, one of which is easily found from $1 + \frac{1}{y^2} = \frac{a}{b^{\frac{1}{2}}}\,\frac{1}{y}$, from what we have previously established. Independently, to find the other root, divide the given equation by $x^{\frac{3}{2}}$ and it becomes

$$x^{\frac{1}{2}} - \frac{a}{x^{\frac{1}{2}}} = -\frac{b}{x^{\frac{3}{2}}}$$

Put $x^{\frac{1}{2}} = ya^{\frac{1}{2}}$ and $x^{\frac{3}{2}} = y^3\,a^{\frac{3}{2}}$;

the equation is thus reduced to

$$y - \frac{1}{y} = -\frac{b}{a^2}\,\frac{1}{y^3}$$

To render the right-hand member of this last equation positive, assume $y = -\frac{1}{z}$, then

$$-\frac{1}{z} + z = +\frac{b}{a^2}\,z^3$$

$$\text{or } 1 - \frac{1}{z^2} = \frac{b}{a^2}\,z^2$$

$$\therefore \downarrow,\left(1 - \frac{1}{z^2}\right) = \downarrow,\left(\frac{b}{a^2}\right) + 2\downarrow,(z).$$

$$\downarrow,\left(\frac{b}{a^2}\right) = {}'167971584 = \downarrow,(c).$$

If $\downarrow,(c) + 2\downarrow,(z)$ be put $= \downarrow,(1) = 0$, we have

$$\downarrow,(z) = 83985792, \quad \therefore z = 2{\cdot}31603550$$

Whence $z$ exists between the limits 1 and 2·3160355, and $\frac{1}{z^2}$ may be put under the form $\frac{1}{4}\downarrow'u$,

$$\therefore \quad {\cdot}f = {\cdot}25000000$$

$$\text{and } 1 - {\cdot}f = {\cdot}75000000$$

$\downarrow, (c) + \downarrow, \left(\frac{1}{\cdot f}\right) - \downarrow, (1 - \cdot f) = \text{'}573924$; which shows that $'k_2$ is required in finding a suitable value for $\downarrow' u_2$.

$$-\frac{f_2}{1-\cdot f} + {'k_2} = -\ \text{'}333333\cdot + \text{'}1005034$$

$$\therefore {'u_2} = \text{'}1 = \frac{\downarrow, (c) + \downarrow, \left(\frac{1}{\cdot f}\right) - \downarrow, (1 - \cdot f)}{-\frac{f_2}{1-\cdot f} + {'k_2}}.$$

$$\begin{array}{rl} \text{Whence,}\quad 25000000 & \\ 250000 & \\ \hline \cdot 24750000 & = \cdot f \\ \cdot 75250000 & = 1 - \cdot f \end{array}$$

and $\downarrow, (c) + \downarrow, \left(\frac{1}{\cdot f}\right) - \downarrow, (1 - \cdot f) = 98309$, which indicates that $k_{3,}$ must be employed to find the value of $u_{3,}$ which is of the ascending branch, since 98309, is an ascending dual logarithm.

$$+\frac{f_2}{1-\cdot f} + k_{3,} = \text{'}32890 + 99950, = 67060,$$

$$\begin{array}{rrl} 67060\cdot) & 98309\cdot & (1, \\ & \underline{67060} & \\ & 31246 & (4, \\ & \underline{26824} & \\ & 4425 & (6, \\ & \underline{4024} & \\ & 401 & (6, \\ & \underline{402} & \end{array}$$

$\therefore \frac{1}{x^2} = \frac{1}{4} \downarrow 0\ \text{'}1\ 1, 4, 6, 6, u_{7,}\ u_{8,}$ the next step may be employed independently of the previous work to find the value of $\frac{1}{x^2}$ true to fourteen consecutive dual digits or exact to fourteen places of decimals; however, the next step gives $u_{7,} = 2$, and $u_{8,} = 4$, :

$$\text{But}\quad x = a\frac{1}{x^2},\ \text{and}\ \downarrow, \left(\frac{1}{x^2}\right) = \text{'}139487897;$$

$$\downarrow, (a) = \cdot 753186701,\ \therefore \downarrow, (x) = 613698804,$$

$$\therefore x = 462\cdot 657948.$$

*Ex.* 4.

When $s$ represents the sine of an arc, $a$, to radius 1, then $7s - 56s^3 + 112s^5 - 64s^7 =$ sine of $(7a)$; if $7a = 180°$, $7a = 360°$, $7a = 540°$, &c., the equation becomes

$$(2^2s^2)^3 - 7\,(2^2s^2)^2 + 14\,(2^2s^2) - 7 = 0;$$

See 'Dual Arithmetic, a new Art,' Part II., pp. 174-177.

Putting $z$ for $2^2s^2$ the equation becomes

$$z^3 - 7z^2 + 14z - 7 = 0.$$

If $x + \frac{7}{3}$ be substituted for $z$ the equation becomes

$$x^3 - \frac{7}{3}\,x = -\frac{7}{27}.$$

A general solution of any equation of the form $x^3 - ax = -b$ may be effected thus:—divide by $x^{\frac{3}{2}}$ and the equation becomes

$$x^{\frac{3}{2}} + \frac{b}{x^{\frac{3}{2}}} = \frac{a}{x^{\frac{1}{2}}};$$

putting $x^{\frac{3}{2}} = yb^{\frac{1}{2}}$ we have $x^{\frac{1}{2}} = y^{\frac{1}{3}}\,b^{\frac{1}{6}}$, and

$$y + \frac{1}{y} = \frac{a}{b^{\frac{2}{3}}}\,\frac{1}{y^{\frac{1}{3}}}$$

$$\therefore \downarrow,\left(1 + \frac{1}{y^2}\right) = \downarrow,\left(\frac{a}{b^{\frac{2}{3}}}\right) - \tfrac{4}{3}\downarrow,(y).$$

$$\downarrow,\left(\frac{a}{b^{\frac{2}{3}}}\right) = 174724901, \quad = \downarrow,(c)$$

*To find the limits.*

If $\downarrow,(c) - \frac{4}{3}\downarrow,(y) = 0$, $\downarrow,(y) = 131043676$,

and $y = 3{\cdot}70779268$;

Again, if $\downarrow,(c) - \frac{4}{3}\downarrow,(y) = \downarrow,(2)$, $\downarrow,(y) = 79057637$,

and $y = 2{\cdot}20466666$.

Then if $L$ be taken = any number whatever between these limits, $\frac{1}{y^2}$ is found by the general formulæ, in a direct manner, and under the form

$$\frac{1}{y^2} = \frac{1}{L^2}\downarrow u_1\,u_2\,u_3\,\ldots\ldots$$

in which $u_1\ u_2\ u_3\ \ldots$ are either ascending or descending dual digits, but in the lowest terms.

$$\text{If } L = 3\cdot \text{ then } \frac{1}{L^2} = \frac{1}{9};$$

$$\cdot f = \cdot 111111111$$

$$1 + \cdot f = 1\cdot 11111111$$

$$\text{Hence, } y^2 = \frac{1}{\cdot f}\,\frac{1}{\downarrow, u}$$

$$\therefore 2 \downarrow, (y) = \downarrow, \left(\frac{1}{\cdot f}\right) - \downarrow, u$$

$$\text{and } \tfrac{4}{3} \downarrow, (y) = \tfrac{2}{3} \downarrow, \left(\frac{1}{\cdot f}\right) - \tfrac{2}{3} \downarrow, u.$$

Consequently,

$$\downarrow, (1 + \cdot f \downarrow u) = \downarrow, (c) - \tfrac{2}{3} \downarrow, \left(\frac{1}{\cdot f}\right) + \tfrac{2}{3} \downarrow, u$$

Then according as we find $u$ to be an ascending or descending dual digit in the position $p$, we have

$$\downarrow, (1 + \cdot f) \pm \frac{f_p \times u_p}{1 + \cdot f} = \downarrow, (c) - \tfrac{2}{3} \downarrow, \left(\frac{1}{\cdot f}\right) + \tfrac{2}{3} k_p \times u_p$$

$$\therefore {}'u_p = \frac{\downarrow, (c) - \tfrac{2}{3} \downarrow, \left(\frac{1}{\cdot f}\right) - \downarrow, (1 + \cdot f)}{\pm \frac{f_p}{1 + \cdot f} - \tfrac{2}{3} k_p},$$

which expression is general in all such equations of the proposed form. In the present instance

$$\downarrow, (c) - \tfrac{2}{3} \downarrow, \left(\frac{1}{\cdot f}\right) - \downarrow, (1 + \cdot f) = 17707210,$$

which indicates that $u_p$ is a descending dual digit in the first position. Now to find its value

$$- \tfrac{2}{3} k_p = - {}'10536052 \times \tfrac{2}{3} = + 7024035.$$

$$- \frac{f_p}{1 + \cdot f} = - \frac{1111111\cdot}{1\cdot 11111111} = - 1000000.$$

$$\overline{6024035.}$$

$\frac{17707210\cdot}{6024035\cdot} = {}'3$ as the most convenient digit for the first position.

$$\left.\begin{array}{r} \cdot 1111111111 + \\ 333333333 - \\ 33333333 + \\ 1111111 - \end{array}\right\} = \cdot f \downarrow {}'3$$

$$\overline{\cdot 0810000000}$$

Then putting $\cdot f = \cdot 081$ and $1 + \cdot f = 1\cdot 081$ we have

$$\downarrow,(c) - \tfrac{2}{3}\downarrow,\left(\frac{1}{\cdot f}\right) - \downarrow,(1+\cdot f) = '617494 = -617494$$

which shows that the next digit is an ascending one in the second position; to find the value of $u_p$ in this case we have

$$+\frac{f_p}{1+\cdot f} - \tfrac{2}{3}k_p = \frac{81000\cdot}{1\cdot 081} - \tfrac{2}{3}(995033,) = -588425$$

$\frac{-617494}{-588425}$ gives 1, for the second digit of the development

$$\begin{array}{r} 81000000 \\ 810000 \\ \hline 81810000 \end{array}$$

$$\therefore \cdot f = \cdot 08181 \text{ and } 1+\cdot f = 1\cdot 08181$$

$$\downarrow,(c) - \tfrac{2}{3}\downarrow,\left(\frac{1}{\cdot f}\right) - \downarrow(1+\cdot f) = '29040$$

which shows that the next digit is ascending and in the fourth position.

$$\frac{f_4}{1\cdot 08181} - \tfrac{2}{3}k_4 = \frac{818\cdot}{1\cdot 8181} - \tfrac{2}{3}(10000,) = -5909;$$

$$\text{and } \frac{-29040}{-5909} \text{ gives } 5,$$

$$\left.\begin{array}{r|r|l} 8181 & 0000 & 0 \\ 4 & 0905 & 0 \\ & 8 & 2 \end{array}\right\} \downarrow^4 5,$$
$$\overline{8185\ 0913\ 2}$$

$$\therefore \cdot f = \cdot 08185091 \text{ and } 1+\cdot f = 1\cdot 08185091;$$

and $\downarrow,(c) - \tfrac{2}{3}\downarrow,\left(\frac{1}{\cdot f}\right) - \downarrow,(1+\cdot f) = 509$, which shows that the remaining digits are of the descending branch and in the 6th, 7th, and 8th positions.

$$-\frac{f_6}{1+\cdot f} - \tfrac{2}{3}\,'k_6 = +59\cdot 09$$

$$\text{and } \frac{509}{59\cdot 09} = '8, '6\ '1$$

$$\therefore \frac{1}{y^2} = \tfrac{1}{9}\downarrow '3\ 1, 0, 5, 0, '8. '6\ '1$$

$$\therefore \downarrow,(y^2) = 250286441,$$

But $x^3 = y^3 b$ and as $\downarrow, (b) = {}^{\prime}134992672$

$\downarrow, (x) = 38431256$, and $\therefore x = 1{\cdot}46860437$

$x + \frac{7}{3} = 3{\cdot}80193770 = (2s)^2 \therefore s = {\cdot}97492791$

Consequently sine of $\frac{540^0}{7} = {\cdot}97492791$

*Ex.* 5.

It is well known that if the function $\frac{x}{\epsilon^x - 1}$ be developed in a series the expression assumes the form

$$\frac{x}{\epsilon^x - 1} = 1 - \frac{x}{2} + B_1 \frac{x^2}{1 \cdot 2} - B_3 \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots ;$$

$B_1$; $B_3$; $B_5$; &c. being the numbers of Bernouilli. Then if the sum of the series, the right-hand member of the above equation, be put $=$ Q, the equation becomes

$$\frac{x}{\epsilon^x - 1} = Q,$$

which may be solved as follows

$\downarrow, (x)$ expresses the dual logarithm of $x$; and $\frac{\downarrow, (x)}{10^8} =$ the hyperbolic log of $x$, written $\downarrow_\epsilon (x)$. In the same manner the logarithm of $m$ to the base $n$ is written $\downarrow_n (m)$.

Now $x$ is the logarithm of $\epsilon^x$ to the base $\epsilon$, then if $z$ be put for $\epsilon^x$, we have $\frac{\downarrow_\epsilon (z)}{z - 1} = Q \quad \therefore \downarrow_\epsilon (z) = Qz - Q \quad \therefore Q = Qz - \downarrow_\epsilon (z)$. Then as a number $q$ may be found, such that

$$\downarrow_\epsilon (q) = Q, \therefore q = \frac{q^z}{z} \text{ or } qz = q^z;$$

$$\therefore \frac{1}{qz} = \frac{1}{q^z}.$$

Taking the $z$ root of both sides of this equation, we have

$$\left(\frac{1}{qz}\right)^{\frac{1}{z}} = \frac{1}{q}$$

again extracting the $q$ root of both sides we obtain

$$\frac{1}{\frac{1}{q^q}} = \left(\frac{1}{qz}\right)^{\frac{1}{qz}}$$

which equation may be put under the form $y^y = b$ and solved in a general manner, which was shown for the first time in 'Dual Arithmetic, a New Art,' pp. 91-102.

$$\frac{1}{q^x} \text{ being determined and put } = A\,;\ z = \epsilon^x = \frac{1}{qA},$$

$$\therefore\ x \downarrow, (\epsilon) = \downarrow, \frac{1}{qA} \text{ and } x = \frac{\downarrow, \left(\frac{1}{qA}\right)}{\downarrow, (\epsilon)}.$$

PROBLEM.

Given the length of the radius P C, to find the angle C P E when the logspiral arc E A and the circular arc D C subtending the angle C P E are equal.

Describe a circular arc A B and draw B F perpendicular to P B meeting the spiral tangent E F in the point F, then the logspiral tangent E F = the spiral arc A E = the circular arc C D by hypothesis.

Let $PA = a^{\phi} = r_1$; and $PE = a^{\theta} = r$; then the length of the logspiral arc $AE = (r - r_1)\left(1 + \frac{1}{(\downarrow_e a)^2}\right)^{\frac{1}{2}}$;

$$\text{but } r_1 = 1, = PA = a^0,$$

$$\text{in this case } \phi = 0.$$

$$\therefore \text{ Spiral arc } AE = (a^{\theta} - 1)\left\{1 + \frac{1}{(\downarrow_e a)^2}\right\}^{\frac{1}{2}}.$$

$\downarrow, (a)$ or $\downarrow, a$ expresses the dual logarithm of $a$;

and $\frac{\downarrow, a}{10^8}$ = the hyperbolic log of $a$, written $\downarrow_e a$.

The logarithm of $m$ to the base $n$ is written $\downarrow_n (m)$.

Putting R = the radius P C, we have $1 : \theta :: R :$ the length of the arc D C, but C D = A E = the straight line E F, whence

$$R\theta = (a^{\theta} - 1)\left\{1 + \frac{1}{(\downarrow_e a)^2}\right\}^{\frac{1}{2}} = (a^{\theta} - 1)\,\frac{\{(\downarrow_e a)^2 + 1\}^{\frac{1}{2}}}{\downarrow_e a}$$

$$\therefore\ \frac{\theta}{a^{\theta} - 1} = \frac{\{(\downarrow_e a)^2 + 1\}^{\frac{1}{2}}}{R \downarrow_e a}, \text{ which put } = Q.$$

Now $\theta$ is the logarithm of $a^{\theta}$ to the base $a$, then if $z$ be put for $a^{\theta}$, we have,

$$\frac{\downarrow_a (z)}{z - 1} = Q.$$

$$\therefore \downarrow_a (z) = Qz - Q$$
$$\therefore Q = Qz - \downarrow_a (z)$$

Then as a number $q$ may be found, such that

$$\downarrow_a q = Q, \text{ therefore,}$$
$$q = \frac{q^z}{z} \text{ or } qz = q^z;$$
$$\therefore \frac{1}{qz} = \frac{1}{q^z}.$$

Taking the $z$ root of both sides of this equation we have

$$\frac{1}{q} = \left(\frac{1}{qz}\right)^{\frac{1}{z}}$$

Again extracting the $q$ root of both sides we obtain

$$\frac{1}{q^{\frac{1}{q}}} = \left(\frac{1}{qz}\right)^{\frac{1}{qz}}$$

which equation may be put under the form $x^x = b$, and solved in a general manner, as is shown in 'Dual Arithmetic,' Part II.

$$\frac{1}{qz} \text{ being determined and put } = A,$$

$$z = a^\theta = \frac{1}{qA}$$
$$\therefore \theta \downarrow, a = \downarrow, \left(\frac{1}{qA}\right)$$
$$\text{and } \theta = \frac{\downarrow, \frac{1}{qA}}{\downarrow, a}.$$

Hence we may find any number of straight lines equal in length to circular and logspiral arcs without the use of series or methods of approximation.

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